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On the Non-Zero Divisor Graphs of Some Finite Commutative Rings

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Abstract

The study of rings and graphs has been explored extensively by researchers. To gain a more effective understanding on the concepts of the rings and graphs, more researches on graphs of different types of rings are required. This manuscript provides a different study on the concepts of commutative rings and undirected graphs. The non-zero divisor graph, $\Gamma(R)$ of a ring R is a simple undirected graph in which its set of vertices consists of all non-zero elements of R and two different vertices are joint by an edge if their product is not equal to zero. In this paper, the commutative rings are the ring of integers modulo n where $n = 8k$ and $k \leq 3$. The zero divisors are found first using the definition and then the non-zero divisor graphs are constructed. The manuscript explores some properties of non-zero divisor graph such as the chromatic number and the clique number. The result has shown that $\Gamma(\mathbb{Z}_{8k})$ is perfect.

Keywords: commutative rings; non-zero divisor graphs; ring; zero divisors.

1 Introduction

In abstract algebra, the theory of rings has its origin in the early 19th century when the commutative and non-commutative rings are being explored. A ring is one of the fundamental algebraic structures consisting of a set with two binary operations which are addition and multiplication [5]. Several elementary results on the concept of rings and groups can be seen in Cohn [5], Auslander and Buchsbaum [2], Rotman [11], Bourbaki [4], Mudaber *et al.* [6] and Romdhini *et al.* [10].

In 1988, the zero-divisor graph was first introduced by Beck [3], where the main focus of the research was on the graph coloring. A few decades later, the concept was further explored by researchers by using different types of groups, rings and fields. Based on Beck's work in [3], Redmond [8] defined a simplified version of the zero-divisor graphs. The author only considered the non-trivial zero divisors of the ring as the vertices of the graph instead of taking all non-zero elements of the ring. In the following year, Redmond [9] explored the ideal-based zero divisor graph of a commutative ring. This study reflects another structure of the subset of zero divisors in a ring since total zero divisor graph engages both ring operations, namely addition and also multiplication.

In 2020, Singh and Bhat [12] surveyed the most recent developments in describing the structural properties of zero-divisor graphs of finite commutative rings and their applications. The idempotent graph of an abelian Rickart ring has been studied by Patil and Momale [7]. The authors proved that the idempotent graph associated to zero divisor graph of an abelian Rickart ring is weakly perfect.

This research is an extension to the concepts of graphs associated with rings. In this study, the concepts of some commutative rings of order $8k$ are being extended to the graph theory, specifically to non-zero divisor graphs. The main focus is to construct the non-zero divisor graphs. The manuscript begins by studying the concepts on rings and graphs. The ring of integers modulo n where $n = 8k$ and $k \leq 3$ is considered. The zero divisors of the rings are found first. Then, the non-zero divisor graphs are constructed. The graph-theoretic properties for the graph, which are the clique number and chromatic number are analyzed and discussed. The perfectness of the graph are investigated.

2 Preliminaries

This section is dedicated to present some fundamental definitions and properties that are related to the main topic. First, the formal definition of the zero divisors of a finite ring is provided as follows.

Definition 2.1. *Zero divisors of a ring are the elements that have product zero when multiplied with each other. If p and q are two non-zero elements of a ring R such that $pq = 0$, then p and q are divisors of 0.*

Graphs associated to groups and rings have been studied extensively by researchers to study the algebraic properties of the groups and rings. Some definitions related to graph theory are presented in the following.

Definition 2.2. *A finite graph, denoted as Γ, is an object with two sets, which are the edge set, $E(\Gamma)$ and the non-empty vertex set, $V(\Gamma)$. The $E(\Gamma)$ may be empty, but otherwise its elements are two-element subsets of the vertex set.*

The concepts of non-zero divisor graph is inspired by the zero divisor graph which is defined as a simple graph with vertices of all zero divisors of a ring R such that two different elements p and q will have an edge between them if and only if $pq = 0$, [1]. The authors studied the interplay between the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma(R)$. The definition on the non-zero divisor graph is given as follows.

Definition 2.3. *The non-zero divisor graph of a ring R, denoted by $\Gamma(R)$, is a simple graph with its vertices are all non-zero elements of a ring such that two distinct elements x and y are adjacent if and only if $xy \neq 0$.*

In the final part of this manuscript, two graph properties namely the chromatic number and clique number are determined based on the constructed non-zero divisor graph. Therefore, definitions of clique number and chromatic number are given as follows.

Definition 2.4. *The minimum amount of colors needed to color the vertices of Γ so that no two neighboring vertices share the same color is the chromatic number and it is usually denoted by $\chi(\Gamma)$.*

Definition 2.5. *Clique is a complete subgraph in a graph, Γ_{sub}. Clique number is the greatest size of a clique in an undirected graph, Γ. It is usually denoted by $\omega(\Gamma)$.*

In addition, the graphs in this manuscript are found to be perfect. The definition of a perfect graph is formally given as follows.

Definition 2.6. *A perfect graph is a graph Γ for which every induced subgraph of Γ has chromatic number equal to its clique number.*

3 Results and Discussions

In this section, the zero divisors of the rings are found. The commutative rings that are being considered in this study are the ring of integers modulo n where $n = 8k$ and $k \leq 3$. The non-zero divisor graphs are then constructed for each ring.

3.1 The zero divisors of ring of integers modulo $8k$

The zero divisors are determined and calculated for the ring of integers modulo $n = 8k$ where $k \leq 3$ by using Definition 2.1. The proposition and proofs are discussed as follows.

Proposition 3.1. *Let $\mathbb{Z}_8$ be the ring of integers modulo eight. The ring $\mathbb{Z}_8$ has three zero divisors and 44 pairs of elements where the multiplication of these elements is not equal to zero.*

Proof. The order of the ring $\mathbb{Z}_8$ is eight since the elements of the ring are $0, 1, 2, 3, 4, 5, 6$ and 7. By Definition 2.1, the zero divisors of $\mathbb{Z}_8$ are determined. The zero divisors of $\mathbb{Z}_8$ are $2, 4$ and 6. There are only three zero divisors found. The set consisting of pairs of elements that have product zero is found and given as follows:

$$\{(p, q)|pq = 0\} = \{(4, 2), (2, 4), (4, 4), (4, 6), (6, 4)\}.$$

□

To find the multiplication of elements in the ring that is not equal to zero, the zero divisors need to be excluded from the calculation. Since there are only seven non-zero elements of the ring, the total pairs of elements are $7 \cdot 7 = 49$. Therefore, the total pairs of elements where the multiplication of these pairs of those non-zero elements is not equal to zero is $49 - 5$ (pairs of elements from zero divisors) $= 44$.

This method of finding the zero divisors is repeated for the ring of integers modulo n where $n = 8k$ and $k \leq 3$. The zero divisors and the pairs of elements where the multiplication of the non-zero elements is not equal to zero are given in the following table.

Table 1: Summary on the zero divisor of $\mathbb{Z}_{8k}$ and its pairs of elements that do not have product zero.

Rings	k	Order	Zero Divisors	$pq \neq 0$
$\mathbb{Z}_8$	1	8	3	44
$\mathbb{Z}_{16}$	2	16	7	208
$\mathbb{Z}_{24}$	3	24	15	476

3.2 The construction of non-zero divisor graphs of ring of integers modulo $8k$

The non-zero divisor graphs of the commutative rings are constructed based on the zero divisors found in the previous subsection.

Proposition 3.2. *The non-zero divisor graph of $\mathbb{Z}_8$, $\Gamma(\mathbb{Z}_8)$ is an undirected graph with seven vertices and 19 edges.*

Proof. Based on Definition 2.3, $\Gamma(\mathbb{Z}_8)$ has seven vertices since $\mathbb{Z}_8$ has seven non-zero elements. The vertices of $\Gamma(\mathbb{Z}_8)$ are connected when the multiplication of the elements is not equal to zero. As stated in Table 1, $\mathbb{Z}_8$ has 44 pairs of elements that do not have product zero. Note that the elements $x \cdot x \neq 0$ is not included as the edges of the non-zero divisor graph to prevent any loops. Hence, $\Gamma(\mathbb{Z}_8)$ has 19 undirected edges. The non-zero divisor graph, $\Gamma(\mathbb{Z}_8)$ is shown in Figure 1. □

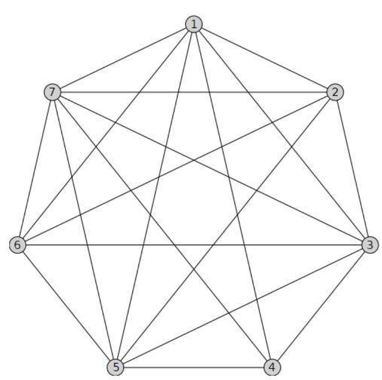

Figure 1: The non-zero divisor graph of $\mathbb{Z}_8$.

It can be seen that the non-zero divisor graphs of $\mathbb{Z}_{8k}$ are undirected graphs. Figure 2 and Figure 3 illustrate the zero divisor graphs of $\mathbb{Z}_{8k}$ where $k = 2$ and 3 respectively.

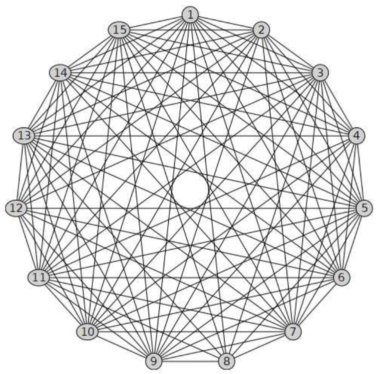

Figure 2: The non-zero divisor graph of $\mathbb{Z}_{16}$.

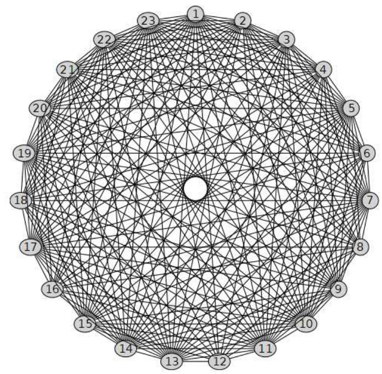

Figure 3: The non-zero divisor graph of $\mathbb{Z}_{24}$.

For the case of $n = p$, the non-zero divisor graph is a complete graph, K_n since there is no zero divisor in the ring. The only divisors of $\Gamma(\mathbb{Z}_p)$ are 1 and p.

Theorem 3.1. *Let $\Gamma(\mathbb{Z}_p)$ be the non-zero divisor graph of ring of integers modulo p where p is a prime number. Then, $\Gamma(\mathbb{Z}_p)$ is isomorphic to K_{p-1} with exactly $((p-1)(p-2))/2$ edges.*

Proof. Since $\mathbb{Z}_p$ has no zero divisors, every vertex is connected with every other vertex. Therefore, $\Gamma(\mathbb{Z}_p)$ is isomorphic to K_{p-1}. As K_n has $n(n-1)/2$ edges, then K_{p-1} has $(p-1)(p-2)/2$ edges. □

3.3 The properties of non-zero divisor graphs of ring of integers modulo $8k$

In this subsection, two graph properties which are the chromatic number and the clique number are obtained after constructing the non-zero divisor graphs of ring of integers modulo $8k$ where $k \leq 3$. The propositions on chromatic number and clique number of non-zero divisor graph are provided.

Proposition 3.3. *The chromatic number and the clique number of $\Gamma(\mathbb{Z}_8)$ are six.*

Proof. The chromatic number of $\Gamma(\mathbb{Z}_8)$, $\chi(\Gamma(\mathbb{Z}_8))$ is six since there are six colors that can be applied on the vertices of $\Gamma(\mathbb{Z}_8)$ so that two neighboring vertices do not share the similar color. For instance, the vertex 1 and 3 have different colors of vertices since there is an edge between them. The non-zero divisor graph of $\mathbb{Z}_8$ with colors is shown in Figure 4. The set of maximum clique of this graph is $\{1, 2, 3, 5, 6, 7\}$. Therefore, based on Definition 2.6, the clique number of $\Gamma(\mathbb{Z}_8)$, $\omega(\Gamma(\mathbb{Z}_8))$ is also six. □

The method of finding the clique number and chromatic numbers is repeated to all the non-zero divisor graphs constructed in subsection 3.2. Figure 5 and Figure 6 show the non-zero divisor graphs of $\mathbb{Z}_{8k}$ with colors where $k = 2$ and 3. The clique number and chromatic number of $\Gamma(\mathbb{Z}_{16})$ and $\Gamma(\mathbb{Z}_{24})$ are found to be 13 and 17 respectively.

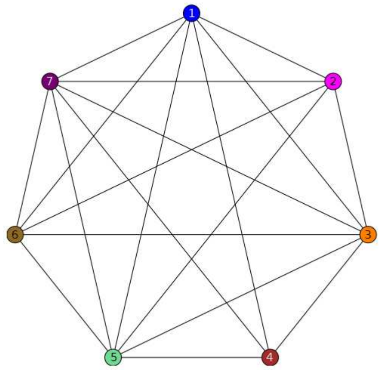

Figure 4: The non-zero divisor graph of $\mathbb{Z}_8$ with colors.

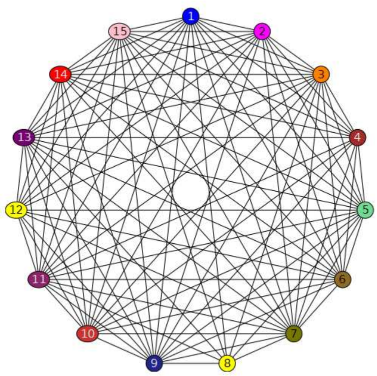

Figure 5: The non-zero divisor graph of $\mathbb{Z}_{16}$ with colors.

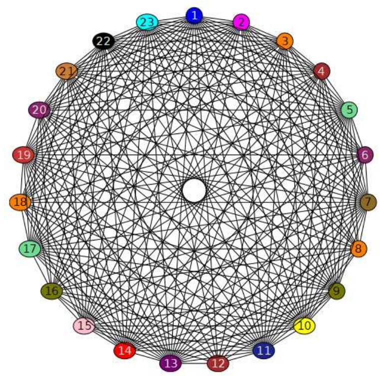

Figure 6: The non-zero divisor graph of $\mathbb{Z}_{24}$ with colors.

Theorem 3.2. *The non-zero divisor graph of the ring of integers modulo $8k$ where $k \leq 3$ is perfect.*

Proof. The clique number and the chromatic number are found first from the non-zero divisor graph as in Proposition 3.3. It can be seen that the chromatic number and clique number of the non-zero divisor graphs are equal. Therefore, by Definition 2.6, the non-zero divisor graphs of $\mathbb{Z}_{8k}$ are perfect since $\omega(\Gamma(\mathbb{Z}_{8k}) = \chi(\Gamma(\mathbb{Z}_{8k}))$. □

Based on Theorem 3.2, the general formula on the perfectness of the non-zero divisor graph of commutative ring can be established.

Theorem 3.3. *The non-zero divisor graph of a commutative ring is perfect.*

Proof. Let $\Gamma(\mathbb{Z}_n)$ be a non-zero divisor graph of a commutative ring of ring of integers modulo n that contains a clique of size q and a valid coloring using q colors. Then, by finding the valid coloring and a clique of size q, it can be seen that $\chi(\Gamma(\mathbb{Z}_n)) \leq q$ and $\omega(\Gamma(\mathbb{Z}_n)) \geq q$. Note that $\omega(\Gamma(\mathbb{Z}_n)) \leq \chi(\Gamma(\mathbb{Z}_n))$ since a new color is needed for each vertex in the largest clique. Therefore, by combining the inequalities, $q \leq \omega(\Gamma(\mathbb{Z}_n) \leq \chi(\Gamma(\mathbb{Z}_n)) \leq q$. Since $\omega(\Gamma(\mathbb{Z}_n) = \chi(\Gamma(\mathbb{Z}_n)) = q$, by Definition 2.6, $\Gamma(\mathbb{Z}_n)$ is a perfect graph. □

4 Conclusions

This manuscript is being made to study the concepts of non-zero divisor graphs on different types of rings. As the continuation of the researches on commutative rings, this study focuses to determine the non-zero divisor graphs of commutative ring, which is the ring of integers modulo n where $n = 8k$ and $k \leq 3$. In conclusion, the zero divisors of $\mathbb{Z}_{8k}$ are found and the non-zero divisor graphs are constructed. The clique number and the chromatic number of the $\Gamma(\mathbb{Z}_{8k})$ are determined and it is found that the non-zero divisor graph of ring of integers modulo n where $n = 8k$ and $k \leq 3$ is perfect.

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Conflicts of Interest The authors declare no conflict of interest.

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